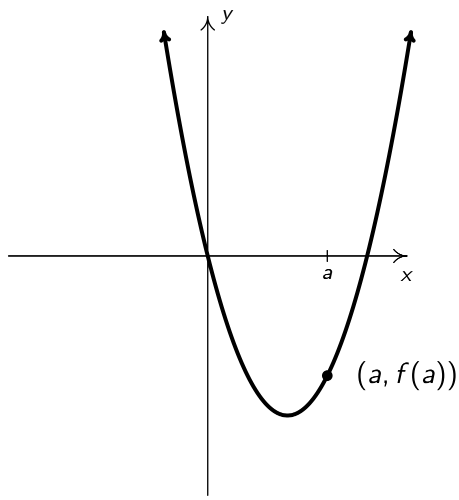
SECTION 2.2: DEFINITIONS OF LIMITS

RECALL FROM COLLEGE ALGBERA: Given a **function** 'f' and a real number 'a' in the **domain** of f, the number $f(a)$ is the **output** from f when the **input** is a. Geometrically, this means $(a, f(a))$ is on the graph of the equation $y = f(x)$. The variable 'x' in the equation '$y = f(x)$' is called the **independent variable** whereas the variable 'y' is called the '**dependent variable**.' More succinctly,

x is the **input** and **independent variable**; $y = f(x)$ is the **output** and **dependent variable**.



$y = f(x)$

EXAMPLE 1: Let $f(x) = \dfrac{x^2 - 2x - 3}{x^2 - 1}$. Attempt to find the following function values. What goes wrong?

- $f(-1)$ Ans: $f(-1) = \dfrac{(-1)^2 - 2(-1) - 3}{(-1)^2 - 1} = \dfrac{0}{0}$ which is undefined.

- $f(1)$ Ans: $f(1) = \dfrac{(1)^2 - 2(1) - 3}{(1)^2 - 1} = \dfrac{-4}{0}$ which is undefined.

Let's look at what's happening graphically... use desmos or some other graphing utility to graph $y = f(x)$. Describe what's going on at:

- $x = -1$ Ans: Things seem (eerily) calm near $x = -1$.

- $x = 1$ Ans: There appears to be a vertical asymptote at $x = 1$.

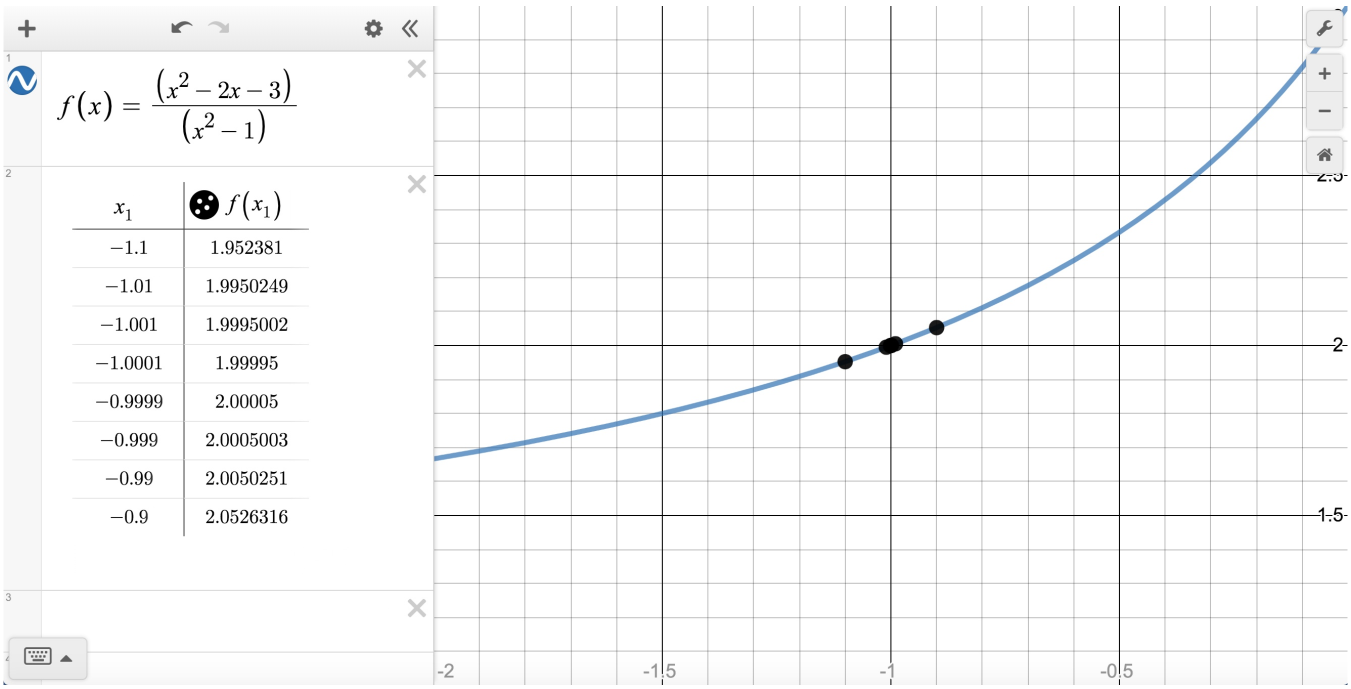
The **BIG IDEA** of Calculus is the concept of **LIMIT**. Informally, the limit concept allows us to investigate functions **near** a given input without worrying about what's actually happening **at** a given input.

(INFORMAL) DEFINITION: The notation '$\lim\limits_{x\to a} f(x)$' is read as 'the limit of $f(x)$ as x approaches a' and is asking what, if anything, the **outputs** $f(x)$ are tending towards as the **inputs** tend towards a.

NOTE: We can 'get a feel' for the value of limits by making 'educated guesses' based on tables and graphs. Don't worry, we'll develop a more formal definition of limit and prove theorems that allow us to determine limits exactly using algebra and related analysis later.

INVESTIGATING LIMITS NUMERICALLY AND GRAPHICALLY: (a.k.a., making educated guesses.)

EXAMPLE 2: Using desmos, we create a table of values of $f(x) = \dfrac{x^2-2x-3}{x^2-1}$ and graph $y = f(x)$ near $x = -1$:



$$f(x) = \frac{\left(x^2-2x-3\right)}{\left(x^2-1\right)}$$

x_1	$f(x_1)$
−1.1	1.952381
−1.01	1.9950249
−1.001	1.9995002
−1.0001	1.99995
−0.9999	2.00005
−0.999	2.0005003
−0.99	2.0050251
−0.9	2.0526316

- Based on the table and graph, as x gets closer and closer to -1, what does $f(x)$ appear to get close to?

Ans: 2

- Make your own table in desmos and sample $f(x)$ for $x-$values even closer to -1. Does your guess change?

Ans: The $f(x)$ values get even closer to 2.

- What is your educated guess as to the value of $\lim\limits_{x\to -1} \dfrac{x^2-2x-3}{x^2-1}$?

Ans: Guess: $\lim\limits_{x\to -1} \dfrac{x^2-2x-3}{x^2-1} = 2.$

EXAMPLE 3 (VIDEO): Use tables and graphs to estimate the following limits.

1. $\lim\limits_{x \to -1} \left(3x^2 - 2x + 1\right)$

   Ans: $\lim\limits_{x \to -1} 3x^2 - 2x + 1 = 6$

2. $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

   Ans: $\lim\limits_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$

3. $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h}$

   Ans: $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h} - 1}{h} = 0.5$

4. $\lim\limits_{t \to 0} \dfrac{\sin(t)}{t}$

   Ans: $\lim\limits_{t \to 0} \dfrac{\sin(t)}{t} = 1$

5. $\lim\limits_{x \to 1} \dfrac{\ln(x)}{x - 1}$

   Ans: $\lim\limits_{x \to 1} \dfrac{\ln(x)}{x - 1} = 1$

6. $\lim\limits_{x \to 1} \dfrac{|x^2 - 1|}{x - 1}$

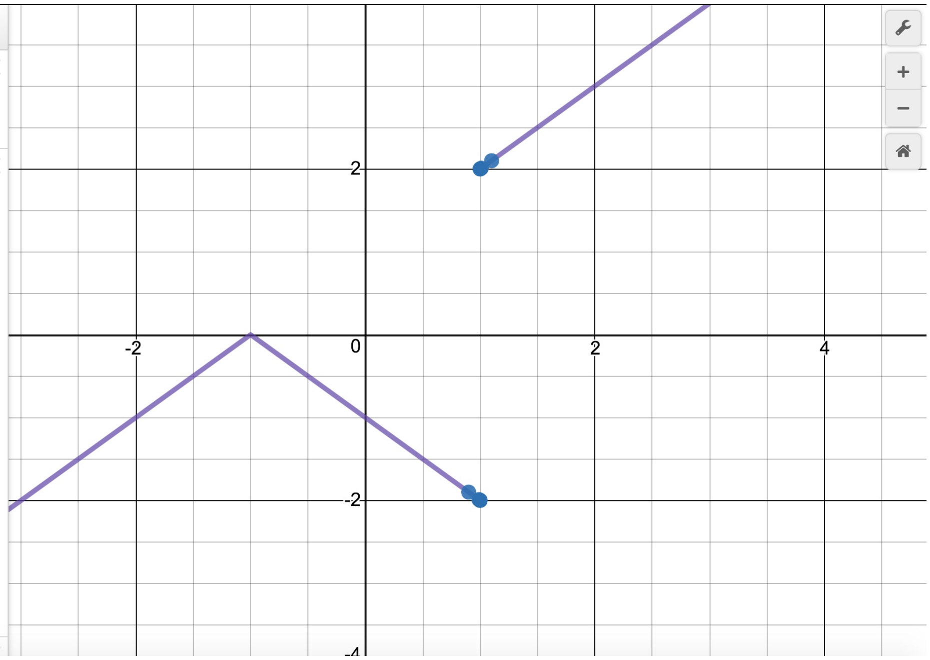
   Ans: If $x < 1$, the function tends towards -2 whereas if $x > 1$, the function tends towards 2. Hmm...

ONE-SIDED LIMITS:

EXAMPLE 4: Let's look a little more closely at the last limit: $\lim_{x \to 1} \frac{|x^2-1|}{x-1}$.

$f(x) = \frac{|x^2-1|}{x-1}$

x_1	$f(x_1)$
0.9	−1.9
0.99	−1.99
0.999	−1.999
0.9999	−1.9999
0.99999	−1.99999
1.00001	2.00001
1.0001	2.0001
1.001	2.001
1.01	2.01
1.1	2.1



Note that:

- When $x < 1$, the values on our table for $f(x)$ are getting closer and closer to -2 as x gets closer to 1.
- When $x > 1$, the values on our table for $f(x)$ are getting closer and closer to 2 as x gets closer to 1.

Since the outputs $f(x) = \frac{|x^2-1|}{x-1}$ approach different values from different directions, we write:

$$\text{`}\lim_{x \to 1} \frac{|x^2-1|}{x-1} \text{ does not exist.'}$$

All that being said, we don't want to lose the fact that if we look from just one direction only, the function behavior is predictable. Hence, we write:

$$\lim_{x \to 1^-} \frac{|x^2-1|}{x-1} = -2 \qquad \text{and} \qquad \lim_{x \to 1^+} \frac{|x^2-1|}{x-1} = 2$$

Here: '$\lim_{x \to 1^-} \frac{|x^2-1|}{x-1}$' is read as 'the limit of $\frac{|x^2-1|}{x-1}$ as x approaches 1 **from the left**.'

Likewise, '$\lim_{x \to 1^+} \frac{|x^2-1|}{x-1}$' is read as 'the limit of $\frac{|x^2-1|}{x-1}$ as x approaches 1 **from the right**.'

TIP: It may help to remember the one-sided notation as follows:

- $x \to 1^-$ means we are sampling values of $x = 1 -$ (a little bit)
- $x \to 1^+$ means we are sampling values of $x = 1 +$ (a little bit)

IN GENERAL: $\lim_{x \to a} f(x) = L$ if, and only if, $\lim_{x \to a^-} f(x) = L$ **and** $\lim_{x \to a^+} f(x) = L$

One-sided limits can come in handy particularly well when analyzing piecewise-defined functions.

EXAMPLE 5: Let $f(x) = \begin{cases} 3x+5 & \text{if } x < 1 \\ x^2 - 4 & \text{if } x \geq 1 \end{cases}$. Determine each of the following, if they exist:

1. $\lim\limits_{x \to 1^-} f(x)$

   HINT: When $x \to 1^-$, we are substituting values for x which are **a little less than** 1 ...

   Ans: $\lim\limits_{x \to 1^-} f(x) = 8$

2. $\lim\limits_{x \to 1^+} f(x)$

   HINT: When $x \to 1^+$, we are substituting values for x which are **a little larger than** 1 ...

   $\lim\limits_{x \to 1^+} f(x) = -3$

3. $\lim\limits_{x \to 1} f(x)$

   HINT: In order for $\lim\limits_{x \to 1} f(x)$ to exist, both $\lim\limits_{x \to 1^-} f(x)$ and $\lim\limits_{x \to 1^+} f(x)$ must exist and **be equal**.

   Ans: $\lim\limits_{x \to 1} f(x)$ does not exist since $\lim\limits_{x \to 1^-} f(x) \neq \lim\limits_{x \to 1^+} f(x)$.

THE FLOOR FUNCTION: Recall the **integers** are the numbers $\{0, \pm 1, \pm 2, \pm 3, ...\}$.

The so-called **floor function**, denoted $f(x) = \lfloor x \rfloor$ or $f(x) = \text{floor}(x)$ is defined as:

$$\lfloor x \rfloor = \text{the largest integer less than or equal to } x.$$

If we break this definition down, we get two cases. If x is an integer, then $\lfloor x \rfloor = x$.

If x is not an integer, $\lfloor x \rfloor$ gives the nearest integer to the **left** of x on the number line.

For some quick examples: $\lfloor 6 \rfloor = 6$, $\lfloor 2.89 \rfloor = 2$, and $\lfloor -116.1 \rfloor = -117$.

NOTE: $f(x) = \lfloor x \rfloor$ is also called the **greatest integer function** and is written $f(x) = \text{int}(x)$ and $f(x) = [|x|]$.

There is also a **ceiling** function as well... can you reason how to define it?

EXAMPLE 6 (VIDEO): Find the following limits, if they exist.

Reason your answer using the definition of the floor function before checking numerically and graphically.

1. $\lim\limits_{x\to 2^+} \lfloor x \rfloor$

Ans: $\lim\limits_{x\to 2^+} \lfloor x \rfloor = 2.$

2. $\lim\limits_{x\to 2^-} \lfloor x \rfloor$

Ans: $\lim\limits_{x\to 2^-} \lfloor x \rfloor = 1.$

3. $\lim\limits_{x\to 2} \lfloor x \rfloor$

Ans: Since $\lim\limits_{x\to 2^+} \lfloor x \rfloor = 2$ but $\lim\limits_{x\to 2^-} \lfloor x \rfloor = 1$, $\lim\limits_{x\to 2} \lfloor x \rfloor = 1$ does not exist.

4. $\lim\limits_{x\to 3^-} x\lfloor x \rfloor$

Ans: $\lim\limits_{x\to 3^-} x\lfloor x \rfloor = 6.$

5. $\lim\limits_{x\to 5^+} \lfloor -x \rfloor$

Ans: $\lim\limits_{x\to 5^+} \lfloor -x \rfloor = -6$

6. $\lim\limits_{x\to -2^-} (x - \lfloor x \rfloor)$

Ans: $\lim\limits_{x\to -2^-} x - \lfloor x \rfloor = 1$

7. $\lim\limits_{x\to -0.1} \dfrac{\lfloor x \rfloor}{x}$

Ans: $\lim\limits_{x\to -0.1} \dfrac{\lfloor x \rfloor}{x} = 10$

8. $\lim\limits_{x\to 0^+} \dfrac{\lfloor x \rfloor}{x}$

Ans: $\lim\limits_{x\to 0^+} \dfrac{\lfloor x \rfloor}{x} = 0$

9. $\lim\limits_{x\to 0^-} \dfrac{\lfloor x \rfloor}{x}$

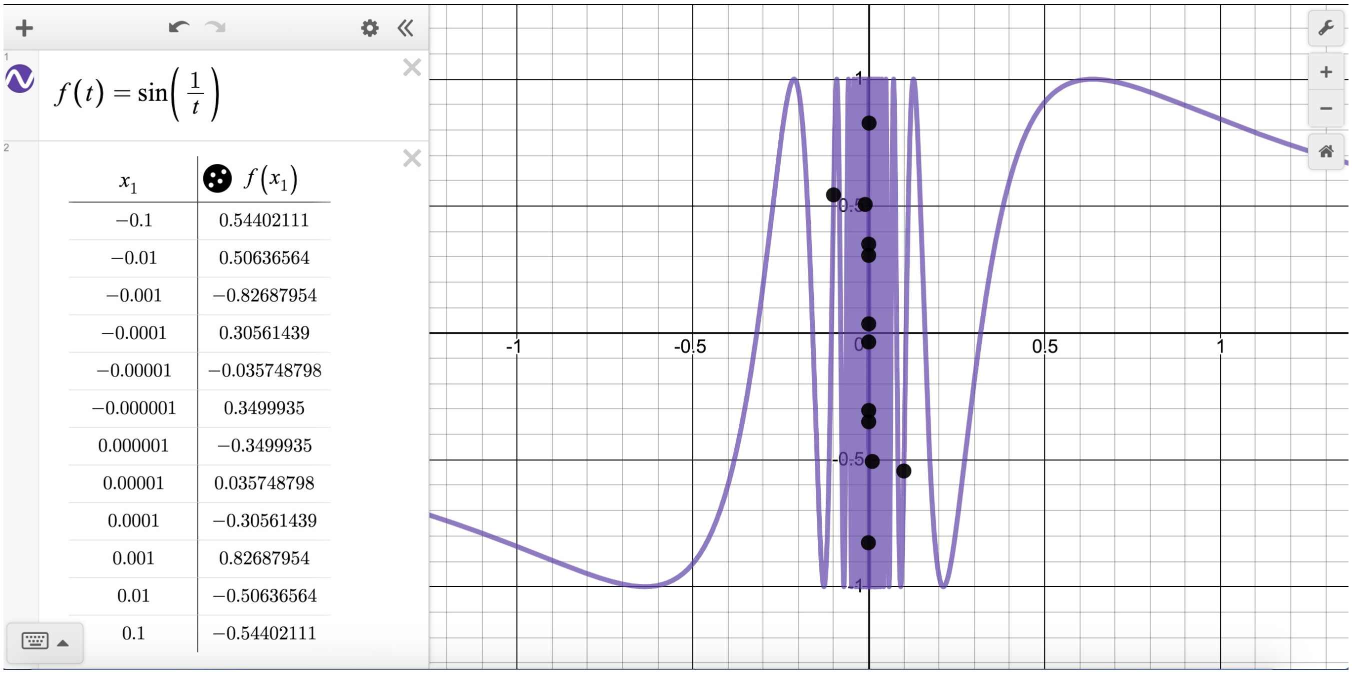
Ans: $\lim\limits_{x\to 0^-} \dfrac{\lfloor x \rfloor}{x}$ does not exist.

AN IMPORTANT TECHNICALITY: Explain why $\lim\limits_{x\to 0} \sqrt{x}$ does not exist but $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.

$\lim\limits_{x\to 0} \sqrt{x}$ does not exist since for $x < 0$, $\sqrt{x}$ is not a real number.

Hence, we must restrict our attention to $x > 0$, and, indeed, $\lim\limits_{x\to 0^+} \sqrt{x} = 0$.

EXAMPLE 7: Consider: $\lim_{t \to 0} \sin\left(\frac{1}{t}\right)$.



x_1	$f(x_1)$
−0.1	0.54402111
−0.01	0.50636564
−0.001	−0.82687954
−0.0001	0.30561439
−0.00001	−0.035748798
−0.000001	0.3499935
0.000001	−0.3499935
0.00001	0.035748798
0.0001	−0.30561439
0.001	0.82687954
0.01	−0.50636564
0.1	−0.54402111

You probably guessed correctly **that** $\lim_{t \to 0} \sin\left(\frac{1}{t}\right)$ does not exist based on the table and the graph.

Let's do a bit of analysis to see **why** the limit doesn't exist. Let's view $f(t) = \sin\left(\frac{1}{t}\right)$ as a composite function.

If we let $x = \frac{1}{t}$, note that as $t \to 0^+$, $x \to \infty$ and, likewise, as $t \to 0^-$, $x \to -\infty$.

We know that the function $f(x) = \sin(x)$ oscillates infinitely many times as $x \to \infty$ and $x \to -\infty$.

Putting this together, as $t \to 0$, $x \to \pm\infty$ so $\sin(x) = \sin\left(\frac{1}{t}\right)$ oscillates infinitely often.

Hence, $\lim_{t \to 0} \sin\left(\frac{1}{t}\right)$ doesn't exist because the outputs from $\sin\left(\frac{1}{t}\right)$ never settle down.

EXAMPLE 8 (VIDEO): Graphs and tables can be deceiving!

1. Use a table and graph to investigate: $\lim_{t \to 0} \sin\left(10^{23} \cdot t\right)$

2. Recall from trigonometry that one cycle of $\sin\left(10^{23} \cdot t\right)$ occurs when $-\pi \leq 10^{23} \cdot t \leq \pi$.

   Solve $-\pi \leq 10^{23} \cdot t \leq \pi$ for t and graph $y = \sin\left(10^{23} \cdot t\right)$ on the resulting window.

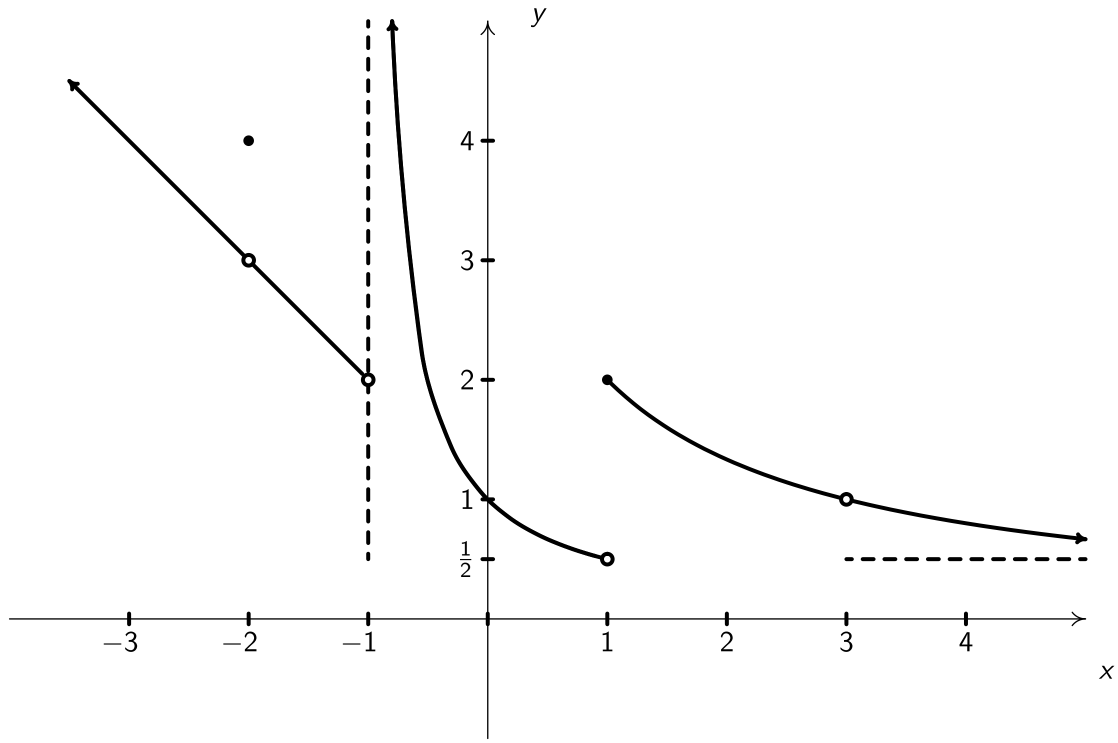
3. Based on your new graph, what is $\lim_{t \to 0} \sin\left(10^{23} \cdot t\right)$?

4. Let $x = 10^{23} \cdot t$ and do some analysis to justify your answer.

QUESTION: How do we know 'how close' to get to a real number 'a' in order to correctly determine $\lim_{x \to a} f(x)$?

ANSWER: We don't ! (?) (Stay tuned - we'll return to this at the end of the chapter . . .)

EXAMPLE 9: Use the graph below to determine the given values.



$y = f(x)$

- $f(-2)$
- $\lim_{x\to -2^-} f(x)$
- $\lim_{x\to -2^+} f(x)$
- $\lim_{x\to -2} f(x)$

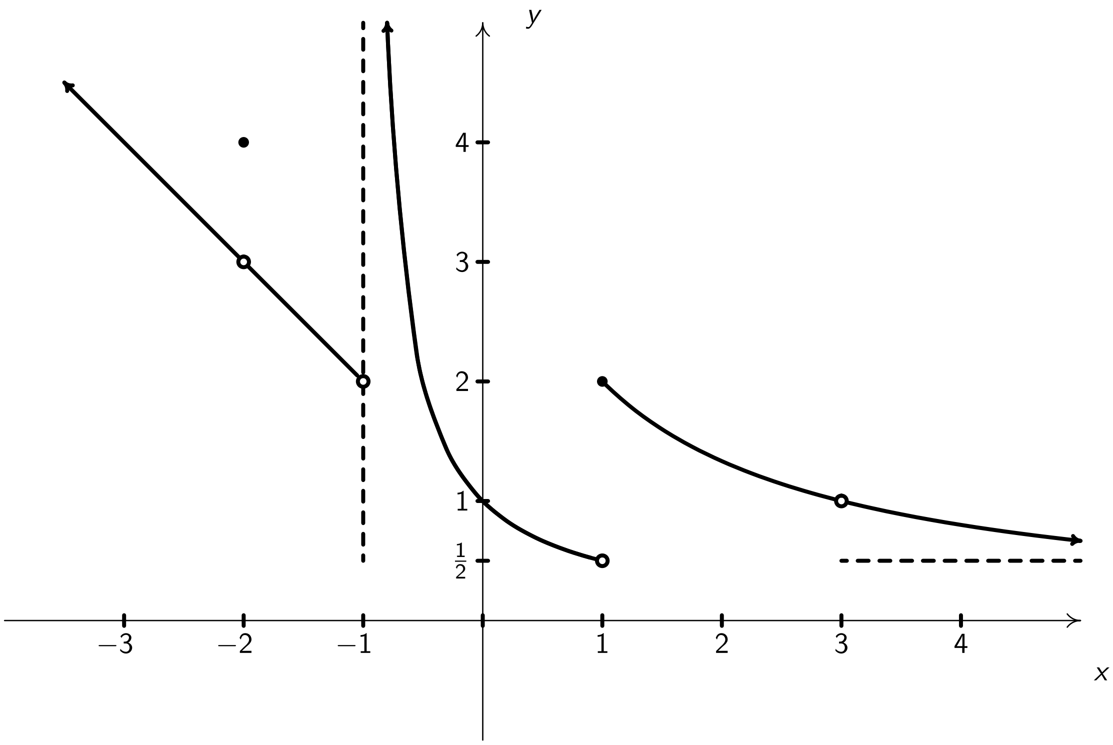
- $f(-1)$
- $\lim_{x\to -1^-} f(x)$
- $\lim_{x\to -1^+} f(x)$
- $\lim_{x\to -1} f(x)$

- $f(0)$
- $\lim_{x\to 0^-} f(x)$
- $\lim_{x\to 0^+} f(x)$
- $\lim_{x\to 0} f(x)$

- $f(1)$
- $\lim_{x\to 1^-} f(x)$
- $\lim_{x\to 1^+} f(x)$
- $\lim_{x\to 1} f(x)$

- $f(3)$
- $\lim_{x\to 3^-} f(x)$
- $\lim_{x\to 3^+} f(x)$
- $\lim_{x\to 3} f(x)$

EXAMPLE 9 (ANSWERS): Use the graph below to determine the given values.



$y = f(x)$

- $f(-2) = 4$
- $\lim_{x \to -2^-} f(x) = 3$
- $\lim_{x \to -2^+} f(x) = 3$
- $\lim_{x \to -2} f(x) = 3$

- $f(-1)$ is undefined.
- $\lim_{x \to -1^-} f(x) = 2$
- $\lim_{x \to -1^+} f(x) = \infty$
- $\lim_{x \to -1} f(x)$ DNE[1]

- $f(0) = 1$.
- $\lim_{x \to 0^-} f(x) = 1$
- $\lim_{x \to 0^+} f(x) = 1$
- $\lim_{x \to 0} f(x) = 1$

- $f(1) = 2$
- $\lim_{x \to 1^-} f(x) = \frac{1}{2}$
- $\lim_{x \to 1^+} f(x) = 2$
- $\lim_{x \to 1} f(x)$ DNE

- $f(3)$ is undefined
- $\lim_{x \to 3^-} f(x) = 1$
- $\lim_{x \to 3^+} f(x) = 1$
- $\lim_{x \to 3} f(x) = 1$

EXAMPLE 10 (VIDEO): Sketch the graph of a function which satisfies all of the following criteria.

- $\lim_{x \to -1^-} f(x) = -3$
- $\lim_{x \to -1^+} f(x) = 1$
- $f(-1) = 1$
- $\lim_{x \to 0} f(x) = -1$
- $f(0)$ is undefined
- $\lim_{x \to 1} f(x) = 0$

HOMEWORK: Section 2.2: 1 - 51 odd, 53*, 55*, 57*

[1] shorthand for 'does not exist.'